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Applying Envelopment Form of Fuzzy DEA to Evaluating the Cost Efficiency of DMUs

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In cost efficiency models, the capability of producing observed outputs of a target decision making unit (DMU) is evaluated by its minimum cost. Traditional cost efficiency models are considered for situations where data set is known for each DMU, while, some of them are imprecise in practice. Several studies have carried out to evaluate cost efficiency using fuzzy data envelopment analysis (DEA) methods for dealing with the imprecise data that have drawbacks. The issue of presenting improve strategy is ignored for inefficient units, as well as the applied models are not easily implemented. This paper proposes a new extension to evaluate fuzzy cost efficiency using fuzzy extended multiplication and division operations. This method offers a fully fuzzy model with triangular fuzzy input-output data along with triangular fuzzy input prices. In the proposed extension, a new definition of fuzzy cost efficiency is suggested based on the extended operations. Finally, a numerical example is provided to show the applicability of the proposed models.
Keywords*: Data Envelopment Analysis; Decision Making Unit; Cost Efficiency; Fuzzy Sets.*

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1. Introduction

In a modern complex environment, with growth and development technology, organizations need to study the performance of their units, such that the strengths and weaknesses of units are recognized to improve the efficiency of the units. DEA is a non-parametric approach for assessing the efficiency and performance of the set of homogenous units. In practice, when the price of input and output for the units is available, units can be evaluated in terms of costs. For company managers, it is important to reduce the cost of their production line without losing their quality. The concept of cost efficiency was first introduced by Farrell [1] and then developed by Färe, Grosskopf and Lovell [2] using linear programming model. Fare et al. measured the cost efficiency of a DMU as the ratio of minimum cost for the production of current outputs with input prices paid by itself to the actually observed cost. In cost efficiency models, outputs of the target DMU are evaluated by the minimum cost. Tone [3] showed that the cost efficiency evaluation model introduced by Fare et al. has several weaknesses. If two DMUs have the same amount of inputs and outputs with the values of different input prices, then the two DMUs have the same cost and technical efficiencies. Using cost-based production possibility set, Tone proposed a new model for evaluation of cost efficiency that overcomes the mentioned shortcoming [3].

The proposed cost efficiency model by Tone [3], considers the situation where input-output data and their corresponding input prices are known exactly for each DMU. In practice, however, the observed values of the input and output data are often imprecise. One of the ways how to take into account the

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uncertainty of the data is the application of fuzzy set theory. Fuzzy set theory, first introduced by Zadeh [4], and has been applied in different real applications, for instance, artificial intelligence, computer science, economics, engineering, management science, operations research, Transportation, information technology, medicine, telecommunication, and robotics [5-15].

Fuzzy DEA concepts were firstly investigated by Sengupta [16] in the fuzzy. Yager [17] has used linear ranking functions to transform fuzzy linear programming problems into the crisp linear programming problems. In recent years, some studies have been carried out to evaluate cost efficiency in the presence of imprecise data in the fuzzy environment.

Jahanshahloo, Hosseinzadeh Lotfi, Alimardani Jondabeh, Banihashemi and Lakzaie [18] initiated cost efficiency assessment underlying fuzzy DEA. They supposed that input prices are specified exactly at each DMU and input-output data are in fuzzy numbers form. Bagherzadeh Valami [19] extended the classical cost efficiency model to a fuzzy environment in order to consider situations where input-output data are known exactly for each DMU and input prices are given in triangular fuzzy numbers form. Paryab, Tavana and Shiraz [20] discussed the fuzzy cost efficiency measurement with both fuzzy input-output data and fuzzy input prices. Moreover, they provided two methods according to the convex DEA and non-convex free disposable hull model with fuzzy variables. Fuzzy multiplication and division operations on fuzzy numbers have not been considered in their proposed models. Puri and Yadav [21] extended the classical cost efficiency and revenue efficiency models to fully fuzzy environments where the input-output data with their corresponding prices are not known exactly. They considered that uncertainty data are triangular fuzzy numbers, and they applied component-wise multiplication on the fuzzy number. Moreover, Pourmahmoud and Bafekr Sharak [22] proposed a new fuzzy model to evaluate the cost efficiency. They applied a family of parametric crisp models to calculate the lower and upper bounds of the α-cut of the fuzzy cost efficiency measures. Furthermore, the problem of ranking DMUs in the fuzzy environment studied based on the cost efficiency.

This paper suggests a new method how to measure fuzzy cost efficiency. Our approach applies fuzzy extended multiplication and division operations in order to introduce a new generalized method. This method proposes fully fuzzy models with triangular fuzzy input-output data along with triangular fuzzy input prices. Unlike previous approaches, the fuzzy extended multiplication and division operations are applied for both fuzzy the cost efficiency measuring and fuzzy cost efficient unit definition. This method is an alternative approach that is based on the extended multiplication and division of fuzzy numbers. The proposed method is an extension of DEA methodology in assessing of cost efficiency of DMUs. Finally, a numerical example is presented to show the applicability and efficiency of the proposed method.

This paper is organized as follows. Section 2 includes a concise introduction to basic notations and definitions of the fuzzy set theory. Moreover, an overview of the Tone's cost efficiency model is outlined in Section 2. Section 3 presents our original fuzzy cost efficiency model. In Section 4, a numerical example illustrating the proposed approach is provided. Finally, conclusions are given in Section 5.

2. Basic concepts

2.1. Fuzzy set theory

In this section, an overview of basic notations and primary definitions of the fuzzy theory is provided [4, 23]. Fuzzy set theory, first introduced by Zadeh [4], has been developed and applied in many real-life problems and still is the subject of research in many areas of operations research and decision making.

Let X be an arbitrary non-empty set, and a crisp subset A of X is defined as a collection of elements $x \in X$ that each x can either belong to or not belong to set A. But for a fuzzy set, an element of X may belong to set A by a degree of membership.

Definition 1: A fuzzy subset $\tilde{A}$ is defined as $\tilde{A} = \{ (x, \mu_{\tilde{A}}(x)); \ x \in X \}$ in set X by membership function $\tilde{A}(x) = \mu_{\tilde{A}}(x) : X \to [0,1]$, where $\mu_{\tilde{A}}(x) \in [0,1]$ demonstrates membership degree of x in $\tilde{A}$.

Definition 2: For any scalar $\alpha \in [0,1]$, α-cut of fuzzy set $\tilde{A}$, denoted by $\tilde{A}_\alpha$, is defined as follows: $\tilde{A}_\alpha = [A^L_\alpha, A^U_\alpha] = \{ x \in X; \mu_{\tilde{A}}(x) \geq \alpha \}$

Definition 3: A fuzzy number $\tilde{A}$ is of LR-type if there are L and R as the reference functions for left and right respectively and the membership function of $\tilde{A}$ is expressed as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} L(\dfrac{m-x}{\alpha}) & ; x \leq m \\ R(\dfrac{x-m}{\beta}) & ; x \geq m \end{cases},$$

where scalars α and β are non-negative. Moreover, the LR-type fuzzy number $\tilde{A}$ is denoted by $(m, \alpha, \beta)_{LR}$, where m is called the mean value of $\tilde{A}$, and α, β are the left and right spreads, respectively. In particular, when $L(x) = R(x) = \begin{cases} 1-x & ; x \geq 0 \\ 0 & ; else \end{cases}$, the fuzzy number $\tilde{A}$, denoted by $\tilde{A} = (m, \alpha, \beta)_{LR} = (m, \alpha, \beta)$ is called a triangular fuzzy number. Note that a triangular fuzzy number $\tilde{A} = (m, \alpha, \beta)$ is non-negative, if $m - \alpha \geq 0$, $\alpha \geq 0$, and $\beta \geq 0$.

A triangular fuzzy number may be demonstrated in another form. A triangular fuzzy number $\tilde{A}$, denoted by form $\tilde{A} = (a^L, a^M, a^U)$, is a fuzzy number in the following membership function:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x - a^L}{a^M - a^L} & ; a^L \leq x \leq a^M \\ \dfrac{x - a^U}{a^M - a^U} & ; a^M \leq x \leq a^U \end{cases},$$

where a^L, a^M, and a^U are lower bound, mean value, and upper bound of fuzzy number $\tilde{A}$, respectively.

Definition 4 (extended multiplication and division):

(1) Let $\tilde{A}=(m,\alpha,\beta)$, and $\tilde{B}=(n,\delta,\gamma)$ be two triangular fuzzy numbers such that m, α, and β are mean value, left spread, and right spread of fuzzy number $\tilde{A}$. Similarly, n, δ, and γ are mean value, left spread, and right spread of fuzzy number $\tilde{B}$. Then extended multiplication, denoted by $\otimes$, is defined as follows:

$$\tilde{A}\otimes\tilde{B}=(m,\alpha,\beta)\otimes(n,\delta,\gamma)=(mn,m\delta+n\alpha,m\gamma+n\beta)$$

Furthermore, extended division, denoted by $\oslash$, is defined as follows:

$$\tilde{A}\oslash\tilde{B}=(m,\alpha,\beta)\oslash(n,\delta,\gamma)=(\frac{m}{n},\frac{m\gamma+n\alpha}{n^2},\frac{m\delta+n\beta}{n^2})$$

(2) Let $\tilde{A}=(a^L,a^M,a^U)$ and $\tilde{B}=(b^L,b^M,b^U)$ be two triangular fuzzy numbers such that a^L, a^M, and a^U are lower bound, mean value, and upper bound of fuzzy number $\tilde{A}$ respectively. Similarly, b^L, b^M, and b^U are lower bound, mean value, and upper bound of fuzzy number $\tilde{B}$ respectively. Then extended multiplication is defined as follows.

$$\tilde{A}\otimes\tilde{B}=(a^Lb^M+b^La^M-a^Mb^M,a^Mb^M,a^Mb^U+b^Ma^U-a^Mb^M)$$

Furthermore, the extended division is defined as follows:

$$\tilde{A}\oslash\tilde{B}=(\frac{a^Lb^M+a^M(\mathrm{b}^M-\mathrm{b}^U)}{(b^M)^2},\frac{a^M}{b^M},\frac{a^Ub^M+a^M(\mathrm{b}^M-\mathrm{b}^L)}{(b^M)^2})$$

In recent years, some researchers have studied for ordering of fuzzy numbers [17, 24-27]. In this paper, the proposed method by Yager [17] is used for ordering of fuzzy subsets. The introduced function by Yager [17] and some properties of this function are provided as follows.

Definition 5: Let $\Im(\Re)$ be the set of all fuzzy numbers. Moreover, suppose $\tilde{A}$ be a fuzzy number, and $[A_\alpha^L,A_\alpha^U]$ be a α-cut of a fuzzy number $\tilde{A}$. Yager [17] defined a linear ranking function as follows.

$$F:\Im(\Re)\to\Re$$

$$F(\tilde{A})=\frac{1}{2}\int_0^1(A_\alpha^L+A_\alpha^U)d\alpha$$

Remark 1: If $\tilde{A}$ be a triangular fuzzy number as form $\tilde{A}=(m,\alpha,\beta)$, then $F(\tilde{A})=m+\frac{1}{4}(\beta-\alpha)$, and if $\tilde{A}$ be as form $\tilde{A}=(a^L,a^M,a^U)$, then $F(\tilde{A})=\frac{1}{4}(a^L+2a^M+a^U)$.

Remark 2: Let $\tilde{A}=(a^L, a^M, a^U)$, and $\tilde{B}=(b^L, b^M, b^U)$ be two triangular fuzzy numbers. Then the relation between fuzzy numbers is defined as follows.

(1) $\tilde{A} \succsim \tilde{B}$, if and only if $F(\tilde{A}) \geq F(\tilde{B})$,

(2) $\tilde{A} \precsim \tilde{B}$, if and only if $F(\tilde{A}) \leq F(\tilde{B})$,

(3) $\tilde{A} \approx \tilde{B}$, if and only if $F(\tilde{A}) = F(\tilde{B})$.

2.2. Cost efficiency

Consider a set of n DMUs and each DMU applies m inputs to produce s outputs. Suppose $x^j=(x_{1j}, x_{2j}, ..., x_{mj})$ and $y^j=(y_{1j}, y_{2j}, ..., y_{sj})$ be the input and output vectors of DMU_j; $j=1,2,...,n$, respectively. Moreover, $w^j=(w_{1j}, w_{2j}, ..., w_{mj})$ be a vector of input prices paid by this DMU. Besides in this study, index j; $j=1,2,...,n$, index i; $i=1,2,...,m$, and index r; $r=1,2,...,s$ are used for DMUs, inputs, and outputs, respectively. The cost efficiency model for a DMU, when the data information is known exactly, introduced by Tone [3] as follows:

$$
\begin{aligned}
C_k^* = \min \quad & \sum_{i=1}^{m} \overline{x}_i \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j \overline{x}_{ij} \leq \overline{x}_i; && \forall i \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rk}; && \forall r \qquad (1)\\
& \lambda_j \geq 0; && \forall j \\
& \overline{x}_i \geq 0; && \forall i,
\end{aligned}
$$

where $\overline{x}_{ij} = w_{ij} x_{ij}$; $\forall i, \forall j$ is i-th input cost of DMU_j., and C_k^* is the minimum cost of producing the observed output of DMU_k. The cost efficiency measure is defined as the ratio of the minimum cost to the actually observed cost of DMU_k.

Traditional methods studied cost efficiency models for situations where input-output data and their corresponding input prices are known exactly at each DMU. In practice, however, the observed values of the input and output data are sometimes imprecise. One of the methods for dealing with uncertain data in DEA is to use of fuzzy DEA models.

Puri and Yadav [21] presented fuzzy models and extended the classical cost efficiency and revenue efficiency models to fully fuzzy environments in a situation where the input-output data along with their corresponding prices were not known exactly. They considered that fuzzy data are triangular fuzzy numbers and applied the component-wise multiplication and division on the fuzzy number.

The extended multiplication and division operation on fuzzy numbers is an alternative method that is applied for solving fuzzy problems. Based on the extended multiplication and division operation, new models are obtained in fuzzy environment. In this paper, the extended method is used for studying the cost efficiency of DMUs which is explained next section.

3. Proposed method

In this section, extensions of the cost efficiency models are presented. Input-output data and prices are supposed to be fuzzy and particularly as triangular fuzzy numbers.

Suppose that the performance of n DMUs is evaluated, and each DMU produce s fuzzy outputs from m fuzzy inputs. Let $\tilde{x}^j = (\tilde{x}_{1j}, \tilde{x}_{2j}, ..., \tilde{x}_{mj})$ be the fuzzy input vector of DMU_j, and $\tilde{y}^j = (\tilde{y}_{1j}, \tilde{y}_{2j}, ..., \tilde{y}_{sj})$ be its fuzzy output vector. Furthermore, assume that $\tilde{w}^j = (\tilde{w}_{1j}, \tilde{w}_{2j}, ..., \tilde{w}_{mj})$ be the vector of fuzzy input prices paid by this DMU.

Under the assumption of constant returns to scale, the minimum fuzzy cost of producing the current output of DMU_k can be measured by the optimal objective function value of the following fuzzy model.

$$
\begin{aligned}
\tilde{R}_k^* \approx \min & \sum_{i=1}^{m} \tilde{\bar{x}}_i \\
s.t. \quad & \sum_{j=1}^{n} \tilde{\lambda}_j \otimes \tilde{\bar{x}}_{ij} \precsim \tilde{\bar{x}}_i ; \quad \forall i \\
& \sum_{j=1}^{n} \tilde{\lambda}_j \otimes \tilde{y}_{rj} \succsim \tilde{y}_{rk} ; \quad \forall r \qquad\qquad (2) \\
& \tilde{\lambda}_j \succsim 0; \quad \forall j \\
& \tilde{\bar{x}}_i \succsim 0; \quad \forall i ,
\end{aligned}
$$

where $\otimes$ denotes the extended multiplication of fuzzy vectors. Also $\precsim$, $\succsim$ are the inequality symbols between fuzzy numbers. Note that $\tilde{\bar{x}}_{ij}$ is the *i*-th input fuzzy cost of DMU_j.

Suppose that the fuzzy data used in the model (2) are in triangular fuzzy numbers form and as $\tilde{A} = (a^L, a^M, a^U)$ where, a^L, a^M, and a^U are lower bound, mean value, and upper bound of fuzzy number $\tilde{A}$ respectively. Let $\tilde{x}_{ij} = (x_{ij}^L, x_{ij}^M, x_{ij}^U)$ be the i-th fuzzy input of DMU_j and $\tilde{y}_{rj} = (y_{rj}^L, y_{rj}^M, y_{rj}^U)$ be its *r*-th fuzzy output. Besides, assume that $\tilde{w}_{ij} = (w_{ij}^L, w_{ij}^M, w_{ij}^U)$ be the vector of fuzzy *i*-th input prices paid by this DMU. In model (2), variables $\tilde{\lambda}_j = (\lambda_j^L, \lambda_j^M, \lambda_j^U)$, and $\tilde{\bar{x}}_i = (\bar{x}_i^L, \bar{x}_i^M, \bar{x}_i^U)$ are considered as non-negative triangular fuzzy numbers and the fuzzy input cost of DMU_j, denoted by $\tilde{\bar{x}}_{ij} = (\bar{x}_{ij}^L, \bar{x}_{ij}^M, \bar{x}_{ij}^U)$, is defined as follows:

$$\tilde{\bar{x}}_{ij} = (\bar{x}_{ij}^{L}, \bar{x}_{ij}^{M}, \bar{x}_{ij}^{U}) = (x_{ij}^{L} w_{ij}^{L}, x_{ij}^{M} w_{ij}^{M}, x_{ij}^{U} w_{ij}^{U}) \tag{3}$$

Therefore, Model (2) is obtained as follows:

$$\begin{aligned}
\tilde{R}_k^* \approx \min & \sum_{i=1}^{m} (\bar{x}_i^{L}, \bar{x}_i^{M}, \bar{x}_i^{U}) \\
s.t. \quad & \sum_{j=1}^{n} (\lambda_j^{L}, \lambda_j^{M}, \lambda_j^{U}) \otimes (\bar{x}_{ij}^{L}, \bar{x}_{ij}^{M}, \bar{x}_{ij}^{U}) \precsim (\bar{x}_i^{L}, \bar{x}_i^{M}, \bar{x}_i^{U}); \quad \forall i \\
& \sum_{j=1}^{n} (\lambda_j^{L}, \lambda_j^{M}, \lambda_j^{U}) \otimes (y_{rj}^{L}, y_{rj}^{M}, y_{rj}^{U}) \succsim (y_{rk}^{L}, y_{rk}^{M}, y_{rk}^{U}); \quad \forall r \\
& (\lambda_j^{L}, \lambda_j^{M}, \lambda_j^{U}) \succsim 0; \quad \forall j \\
& (\bar{x}_i^{L}, \bar{x}_i^{M}, \bar{x}_i^{U}) \succsim 0; \quad \forall i,
\end{aligned} \tag{4}$$

Model (4) is recognized as a fuzzy linear programming problem. Using linear ranking function proposed by Yager [17], the fuzzy objective function is transformed into the crisp objective function. After putting the fuzzy variables in triangular forms, and applying extended arithmetic operations on triangular fuzzy numbers and using relations of fuzzy inequalities, model (4) can be transformed into the following linear model:

$$\begin{aligned}
RC_k = \min & \frac{1}{4} \left(\sum_{i=1}^{m} \bar{x}_i^{L} + 2\sum_{i=1}^{m} \bar{x}_i^{M} + \sum_{i=1}^{m} \bar{x}_i^{U} \right) \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j^{L} \bar{x}_{ij}^{M} - \sum_{j=1}^{n} \lambda_j^{M} (\bar{x}_{ij}^{M} - \bar{x}_{ij}^{L}) \le \bar{x}_i^{L}; \quad \forall i \\
& \sum_{j=1}^{n} \lambda_j^{M} \bar{x}_{ij}^{M} \le \bar{x}_i^{M}; \quad \forall i \\
& \sum_{j=1}^{n} \lambda_j^{U} \bar{x}_{ij}^{M} + \sum_{j=1}^{n} \lambda_j^{M} (\bar{x}_{ij}^{U} - \bar{x}_{ij}^{M}) \le \bar{x}_i^{U}; \quad \forall i \\
& \sum_{j=1}^{n} \lambda_j^{L} y_{rj}^{M} - \sum_{j=1}^{n} \lambda_j^{M} (y_{rj}^{M} - y_{rj}^{L}) \ge y_{rk}^{L}; \quad \forall r \\
& \sum_{j=1}^{n} \lambda_j^{M} y_{rj}^{M} \ge y_{rk}^{M}; \quad \forall r \\
& \sum_{j=1}^{n} \lambda_j^{U} y_{rj}^{M} + \sum_{j=1}^{n} \lambda_j^{M} (y_{rj}^{U} - y_{rj}^{M}) \ge y_{rk}^{U}; \quad \forall r \\
& \bar{x}_i^{M} - \bar{x}_i^{L} \ge 0, \ \bar{x}_i^{U} - \bar{x}_i^{M} \ge 0, \ \bar{x}_i^{L} \ge 0; \quad \forall i \\
& \lambda_j^{M} - \lambda_j^{L} \ge 0, \ \lambda_j^{U} - \lambda_j^{M} \ge 0, \ \lambda_j^{L} \ge 0; \quad \forall j
\end{aligned} \tag{5}$$

In the production process, the input–output vectors specified by DMUs must be technically feasible. It is important, therefore, to determine the feasible input–output vectors for evaluating cost efficiency.

To this end, it is necessary to examine feasibility of the proposed models. The following theorems show the feasibility and boundedness of the model (5).

Theorem 1: The proposed linear model (5) is always feasible, and the its optimal value is bounded.

Proof: In order to prove feasibility of linear programming model (5), it is sufficient to introduce a feasible solution. Consider values for variables of the model (5) as follows.

$$\begin{aligned}
&\lambda_k = 1 \\
&\lambda_j = 0 \qquad ;\forall j,\ j \neq k \\
&\overline{x}_i^L = \overline{x}_{ik}^L \qquad ;\forall i \qquad\qquad (6) \\
&\overline{x}_i^M = \overline{x}_{ik}^M \qquad ;\forall i \\
&\overline{x}_i^U = \overline{x}_{ik}^U \qquad ;\forall i
\end{aligned}$$

Since the values given by relation (6) satisfy to all the constraints of model (5), the assumed solution (6) for different fuzzy input-output values is a feasible solution to the model (5). Therefore, the linear programming model (5) always has at least one feasible solution.

The linear programming model (5) is a minimization problem, so to prove boundedness of the its optimal value, it is just adequate to determine a lower bound for the objective function. Noting that the values of the variables $\overline{x}_i^L, \overline{x}_i^M, \overline{x}_i^U$ are positive, and then the value of the objective function for each feasible solution is positive. Therefore, the optimal value of the objective function is always greater than zero. On the other hand, the variables $\overline{x}_i^L, \overline{x}_i^M, \overline{x}_i^U$ are bounded from below according to the first, second, and third constraints of the model (5). Therefore, the optimal value of the objective function of the assumed model will always be bounded from below. In other words, the optimal value of the model (5) is bounded. This completes the proof. ■

It should be noted that the extended division on fuzzy numbers are used to obtain the fuzzy cost efficiency of decision-making units and to define the fuzzy cost efficiency unit. Moreover, a new definition of cost efficiency is proposed in the following terms:

Definition 6: The fuzzy cost efficiency $\tilde{E}_K^C$ of DMU_k is defined as the ratio of the minimum fuzzy cost to the actual observed fuzzy cost of DMU_k as follows:

$$\tilde{E}_K^C = \sum_{i=1}^m \tilde{\overline{x}}_i^* \oslash \sum_{i=1}^m \tilde{\overline{x}}_{ik} = \left(\sum_{i=1}^m \overline{x}_i^{L*}, \sum_{i=1}^m \overline{x}_i^{M*}, \sum_{i=1}^m \overline{x}_i^{U*}\right) \oslash \left(\sum_{i=1}^m \overline{x}_{ik}^L, \sum_{i=1}^m \overline{x}_{ik}^M, \sum_{i=1}^m \overline{x}_{ik}^U\right) \qquad (7)$$

where $\oslash$ denotes the extended division on fuzzy vectors. Further in the above definition $\tilde{\overline{x}}_i^* = (\overline{x}_i^{L*}, \overline{x}_i^{M*}, \overline{x}_i^{U*});\ \ \forall i = 1,...,m$ is the optimal solution obtained from model (5) and $\tilde{\overline{x}}_{ik}$ is also obtained from relation (3).

Definition 7: DMU_k is the fuzzy cost efficient unit if the minimum fuzzy cost resulted by model (5) be the same the observed fuzzy cost for DMU_k.

Note that the input-output values of a DMU can be measured by different methods. This indicates the difference in the measurement scale for the input-output data of the DMUs. Therefore, the proposed models should be invariant relative to scale changes in order to apply different data for different scales. Thus, it is necessary to examine scale invariance for the proposed models.

Theorem 2: With regard to the invariance properties of the linear model (5) to input-output data, two following statements are satisfied:

1) Model (5) is invariant with respect to the scale change in fuzzy outputs.
2) Model (5) is not invariant with respect to the scale change in fuzzy inputs.

Proof: In the proposed model (5), data consists of the product of the fuzzy input vector and the fuzzy input price vector, denoted by $\tilde{\bar{x}}_{ij}$, and as well as the fuzzy output vector $\tilde{y}_{rj}$. Suppose, in the given model, the data scale of the DMUs is changed, and data is multiplied by scalar quantities as follows:

$$\begin{aligned} \tilde{\bar{x}}'_{ij} &= a_i \tilde{\bar{x}}_{ij} \\ \tilde{y}'_{rj} &= b_r \tilde{y}_{rj} \end{aligned} \tag{8}$$

where $\tilde{x}'_{ij} = \alpha_i \tilde{x}_{ij}$, $\tilde{p}'_{ij} = \beta_i \tilde{p}_{ij}$ and $a_i = \alpha_i \beta_i$. Model (5) is rewritten with new data as follows:

$$\begin{aligned} RC_k = \min\ & \frac{1}{4}\left(\sum_{i=1}^{m} \bar{x}_i^{L} + 2\sum_{i=1}^{m} \bar{x}_i^{M} + \sum_{i=1}^{m} \bar{x}_i^{U}\right) \\ s.t.\quad & \sum_{j=1}^{n} \lambda_j^{L} \bar{x}'^{M}_{ij} - \sum_{j=1}^{n} \lambda_j^{M} (\bar{x}'^{M}_{ij} - \bar{x}'^{L}_{ij}) \le \bar{x}_i^{L}; \quad \forall i \\ & \sum_{j=1}^{n} \lambda_j^{M} \bar{x}'^{M}_{ij} \le \bar{x}_i^{M}; \quad \forall i \\ & \sum_{j=1}^{n} \lambda_j^{U} \bar{x}'^{M}_{ij} + \sum_{j=1}^{n} \lambda_j^{M} (\bar{x}'^{U}_{ij} - \bar{x}'^{M}_{ij}) \le \bar{x}_i^{U}; \quad \forall i \\ & \sum_{j=1}^{n} \lambda_j^{L} y'^{M}_{rj} - \sum_{j=1}^{n} \lambda_j^{M} (y'^{M}_{rj} - y'^{L}_{rj}) \ge y'^{L}_{rk}; \quad \forall r \\ & \sum_{j=1}^{n} \lambda_j^{M} y'^{M}_{rj} \ge y'^{M}_{rk}; \quad \forall r \\ & \sum_{j=1}^{n} \lambda_j^{U} y'^{M}_{rj} + \sum_{j=1}^{n} \lambda_j^{M} (y'^{U}_{rj} - y'^{M}_{rj}) \ge y'^{U}_{rk}; \quad \forall r \\ & \bar{x}_i^{M} - \bar{x}_i^{L} \ge 0,\ \bar{x}_i^{U} - \bar{x}_i^{M} \ge 0,\ \bar{x}_i^{L} \ge 0; \quad \forall i \\ & \lambda_j^{M} - \lambda_j^{L} \ge 0,\ \lambda_j^{U} - \lambda_j^{M} \ge 0,\ \lambda_j^{L} \ge 0; \quad \forall j \end{aligned} \tag{9}$$

By applying the relations (8) in model (9), model (9) can be written as follows:

$$
\begin{aligned}
RC_k = \min \frac{1}{4} & (\sum_{i=1}^{m} \overline{x}_i^{L} + 2\sum_{i=1}^{m} \overline{x}_i^{M} + \sum_{i=1}^{m} \overline{x}_i^{U}) \\
s.t. \quad & a_i(\sum_{j=1}^{n} \lambda_j^L \overline{x}_{ij}^M - \sum_{j=1}^{n} \lambda_j^M (\overline{x}_{ij}^M - \overline{x}_{ij}^L)) \le \overline{x}_i^L ; \quad \forall i \\
& a_i(\sum_{j=1}^{n} \lambda_j^M \overline{x}_{ij}^M) \le \overline{x}_i^M ; \quad \forall i \\
& a_i(\sum_{j=1}^{n} \lambda_j^U \overline{x}_{ij}^M + \sum_{j=1}^{n} \lambda_j^M (\overline{x}_{ij}^U - \overline{x}_{ij}^M)) \le \overline{x}_i^U ; \quad \forall i \\
& b_r(\sum_{j=1}^{n} \lambda_j^L y_{rj}'^{\,M} - \sum_{j=1}^{n} \lambda_j^M (y_{rj}^M - y_{rj}^L)) \ge b_r y_{rk}^L ; \quad \forall r \qquad (10) \\
& b_r(\sum_{j=1}^{n} \lambda_j^M y_{rj}^M) \ge b_r y_{rk}^M ; \quad \forall r \\
& b_r(\sum_{j=1}^{n} \lambda_j^U y_{rj}^M + \sum_{j=1}^{n} \lambda_j^M (y_{rj}^U - y_{rj}^M)) \ge b_r y_{rk}^U ; \quad \forall r \\
& \overline{x}_i^M - \overline{x}_i^L \ge 0, \ \overline{x}_i^U - \overline{x}_i^M \ge 0, \ \overline{x}_i^L \ge 0; \quad \forall i \\
& \lambda_j^M - \lambda_j^L \ge 0, \ \lambda_j^U - \lambda_j^M \ge 0, \ \lambda_j^L \ge 0; \quad \forall j
\end{aligned}
$$

In model (10), the sides of the first, second and third constraints in the scalar $1/a_i\,; a_i \ne 0$ and the sides of the fourth, fifth, and sixth constraints are multiplied by the scalar $1/b_i\,; b_i \ne 0$. Consider substitution variables as follows:

$$\overline{z}_i^L = \frac{\overline{x}_i^L}{a_i}, \quad \overline{z}_i^M = \frac{\overline{x}_i^M}{a_i}, \quad \overline{z}_i^U = \frac{\overline{x}_i^U}{a_i} \qquad (11)$$

By applying the substitution variables, model (10) is written as follows:

$$
\begin{aligned}
RC_k = \min \frac{1}{4} & (\sum_{i=1}^{m} a_i \overline{z}_i^{L} + 2\sum_{i=1}^{m} a_i \overline{z}_i^{M} + \sum_{i=1}^{m} a_i \overline{z}_i^{U}) \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j^L \overline{x}_{ij}^M - \sum_{j=1}^{n} \lambda_j^M (\overline{x}_{ij}^M - \overline{x}_{ij}^L) \le \overline{z}_i^L ; \quad \forall i \\
& \sum_{j=1}^{n} \lambda_j^M \overline{x}_{ij}^M \le \overline{z}_i^M ; \quad \forall i \\
& \sum_{j=1}^{n} \lambda_j^U \overline{x}_{ij}^M + \sum_{j=1}^{n} \lambda_j^M (\overline{x}_{ij}^U - \overline{x}_{ij}^M) \le \overline{z}_i^U ; \quad \forall i \\
& \sum_{j=1}^{n} \lambda_j^L y_{rj}^M - \sum_{j=1}^{n} \lambda_j^M (y_{rj}^M - y_{rj}^L) \ge y_{rk}^L ; \quad \forall r \qquad (12) \\
& \sum_{j=1}^{n} \lambda_j^M y_{rj}^M \ge y_{rk}^M ; \quad \forall r \\
& \sum_{j=1}^{n} \lambda_j^U y_{rj}^M + \sum_{j=1}^{n} \lambda_j^M (y_{rj}^U - y_{rj}^M) \ge y_{rk}^U ; \quad \forall r \\
& \overline{z}_i^M - \overline{z}_i^L \ge 0, \ \overline{z}_i^U - \overline{z}_i^M \ge 0, \ \overline{z}_i^L \ge 0; \quad \forall i \\
& \lambda_j^M - \lambda_j^L \ge 0, \ \lambda_j^U - \lambda_j^M \ge 0, \ \lambda_j^L \ge 0; \quad \forall j
\end{aligned}
$$

Obviously, the constraints of the model (12) are the same constraints of model (5), while the objective function of the two models is different. Therefore, according to the definition of invariance, it can be concluded that the model (5) is not invariant with respect to the scale change in fuzzy input values.
Now, if the scale change occurs only in the values of fuzzy output, then model (12) is the same model (5). Therefore, it can be concluded that the model (5) is invariant with respect to the scale change in the fuzzy output values. This completes the proof. ■

4. Applications

In this section, a numerical example is provided to illustrate the proposed approach – the data set is taken from Puri and Yadav [21]. In this example, 10 DMUs is considered and each of them produces two fuzzy outputs from two fuzzy inputs. The fuzzy data are in triangular fuzzy numbers form. Table 1 presents the fuzzy input-output data and fuzzy input prices for all DMUs. All calculations were performed using GAMS modelling system with CPLEX optimization solver.

Table 1. Fuzzy input-output data and fuzzy input prices for DMUs

DMU	Fuzzy inputs		Fuzzy outputs		Fuzzy input prices	
	Input1	Input2	Output1	Output2	Input1	Input2
1	(49.4,53,56.6)	(40.5,45,49.5)	(70,77.5,85)	(28.8,35.4,42)	(4.7,5,5.3)	(4.5,5,5.5)
2	(20,25,30)	(41.6,46.5,51.4)	(41.5,49.6,57.7)	(28.9,34.7,40.5)	(3.55,6.5)	(4.5,6,7.5)
3	(11.5,18,24.5)	(11.9,15.7,19.5)	(21.6,26.5,31.4)	(29.4,37.6,45.8)	(7.28,8.8)	(6.3,7,7.7)
4	(12.1,18,23.9)	(20.1,25.5,30.9)	(34.2,37.5,40.8)	(40.9,47.5,54.1)	(7.2,9,10.8)	(5,5.5,6)
5	(29.1,32,34.9)	(20.3,25,29.7)	(59.2,64,68.8)	(72.9,76.4,79.9)	(2.8,3,3.2)	(1.75,2,2.25)
6	(50.8,56,61.2)	(41.6,45.1,48.6)	(32.6,35.3,38)	(3742.5,48)	(5.4,6,6.6)	(3.25,4,4.75)
7	(17.6,24,30.4)	(13.6,17.5,21.4)	(75.6,82.9,90.2)	(33.2,38.5,43.8)	(1.6,2,2.4)	(1.5,2,2.5)
8	(71.2,78,84.8)	(18.5,23.9,29.3)	(60,66,72)	(38.2,47.4,56.6)	(1.8,3,4.2)	(3,3.9,4.8)
9	(45.5,52,58.5)	(13.7,19.8,25.9)	(51.1,56.5,61.9)	(51.3,56,60.7)	(4.8,5,5.2)	(89.8,11.6)
10	(44.6,49,53.4)	(16.3,20.6,24.9)	(38,46.5,55)	(32.1,38,43.9)	(2,2.9,3.8)	(2,2.6,3.2)

The performance of the DMUs in terms of costs is evaluated using new method. Based on the results of the evaluation, the fuzzy cost efficiency is derived using (7).
The minimum fuzzy cost for each DMU is in triangular fuzzy form and as $(\sum_{i=1}^{m}\bar{x}_i^{L*},\sum_{i=1}^{m}\bar{x}_i^{M*},\sum_{i=1}^{m}\bar{x}_i^{U*})$ where $\tilde{\bar{x}}_i^{*}=(\bar{x}_i^{L*},\bar{x}_i^{M*},\bar{x}_i^{U*});\quad \forall i$ is the optimal solution given by model (5). Moreover, $\bar{x}_i^{L*}$, $\bar{x}_i^{M*}$, and $\bar{x}_i^{U*}$ are lower bound, mean value, and upper bound of the minimum fuzzy cost respectively. Results derived by applying model (5) are shown in Table 2. Column (2) of Table 2 contains lower bound, mean value, and upper bound of minimum fuzzy cost. The quantities of the observed fuzzy cost are given in column (3) and the values of the fuzzy cost efficiency are presented in column (4).

Table 2. Results of model (5)

DMU	Minimum fuzzy cost	Observed fuzzy cost	Fuzzy cost efficiency
(1)	**(2)**	**(3)**	**(4)**
1	(44.721, 77.593, 120.235)	(414.43, 490, 572.23)	(0.065, 0.158, 0.27)
2	(43.319, 70.133, 101.481)	(257.2, 404, 580.5)	(0.031, 0.174, 0.314)
3	(45.205, 71.853, 100.229)	(157.77, 253.9, 365.75)	(0.053, 0.283, 0.502)
4	(64.209, 90.985, 120.107)	(187.62, 302.25, 443.52)	(0.072, 0.301, 0.512)
5	(117.005, 146, 178.505)	(117.005, 146, 178.505)	(0.579, 1, 1.421)
6	(58.1, 81.729, 107.703)	(409.52, 516.4, 634.77)	(0.076, 0.158, 0.241)
7	(48.56, 83, 126.46)	(48.56, 83, 126.46)	(0.061, 1, 1.939)
8	(56.98, 96.222, 141.98)	(183.66, 327.21, 496.8)	(0.022, 0.294, 0.563)
9	(80.974, 108.853, 140.717)	(328, 454.04, 604.64)	(0.099, 0.24, 0.376)
10	(48.807, 75.348, 105.734)	(121.8, 195.66, 282.6)	(0.078, 0.385, 0.686)

According to the presented results in Table 2 and based on the Definition 7, the units DMU5 and DMU7 are considered as cost efficient units in terms of cost efficiency. For these two efficient units, the upper bound of the fuzzy cost efficiency is greater than one due to the extended division on triangular fuzzy numbers. As shown in Table 2, minimum fuzzy cost and the observed fuzzy cost are equal for efficient units including DMU5 and DMU7. Moreover, other DMUs are inefficient units, and as a result, DMU5 and DMU7 become the reference points of the inefficient units. Table 3 shows the reference set for cost inefficient units.

Table 3. Results of reference units

DMU	Reference Set	Optimal Coefficient	
1	DMU5, DMU7	$\tilde{\lambda}_5^* = (0, 0, 0.014)$,	$\tilde{\lambda}_7^* = (0.927, 0.935, 0.935)$
2	DMU5, DMU7	$\tilde{\lambda}_5^* = (0.23, 0.25, 0.27)$,	$\tilde{\lambda}_7^* = (0.373, 0.405, 0.437)$
3	DMU5	$\tilde{\lambda}_5^* = (0.407, 0.492, 0.577)$	
4	DMU5, DMU7	$\tilde{\lambda}_5^* = (0.553, 0.610, 0.667)$,	$\tilde{\lambda}_7^* = (0.023, 0.023, 0.023)$
6	DMU5, DMU7	$\tilde{\lambda}_5^* = (0.485, 0.529, 0.573)$,	$\tilde{\lambda}_7^* = (0.054, 0.054, 0.054)$
8	DMU5, DMU7	$\tilde{\lambda}_5^* = (0.255, 0.319, 0.383)$,	$\tilde{\lambda}_7^* = (0.598, 0.598, 0.598)$
9	DMU5, DMU7	$\tilde{\lambda}_5^* = (0.616, 0.635, 0.659)$,	$\tilde{\lambda}_7^* = (0.195, 0.195, 0.195)$
10	DMU5, DMU7	$\tilde{\lambda}_5^* = (0.331, 0.351, 0.372)$,	$\tilde{\lambda}_7^* = (0.249, 0.290, 0.331)$

It can be seen that, DMU3 chooses only DMU5, with optimal coefficient values $\tilde{\lambda}_5^* = (\lambda_5^{*L}, \lambda_5^{*M}, \lambda_5^{*U}) = (0.407, 0.492, 0.577)$ as its reference unit. In fact, DMU3 obtains minimum fuzzy cost to produce fuzzy outputs $\tilde{y}_1^* = (23.686,\ 31.488,\ 39.29)$ and $\tilde{y}_2^* = (29.373,\ 37.589,\ 45.805)$ using fuzzy inputs cost $\tilde{\tilde{x}}_1^* = (31.928,\ 47.232,\ 63.107)$ and $\tilde{\tilde{x}}_2^* = (13.228,\ 24.6,\ 37.128)$.

Besides, DMU5 and DMU7 are chosen as a reference set by other inefficient units. For instance, in performance evaluation of DMU1 that is determined as an inefficient unit, DMU5 and DMU7 introduce as its reference set with optimal coefficient $(\lambda_5^{*L}, \lambda_5^{*M}, \lambda_5^{*U}) = (0, 0, 0.014)$ and $(\lambda_7^{*L}, \lambda_7^{*M}, \lambda_7^{*U}) = (0.927, 0.927, 0.935)$. In addition, lower bound, mean value, and upper bound of fuzzy optimal coefficient are equal in DMU7 as reference point for each of four inefficient DMUs, including DMU4, DMU6, DMU8, and DMU9.

According to the obtained results, DMU6 is the worse cost efficient unit that the same conclusion is derived from the data set of Table l. It can be observed that, this unit spends too expensive high input volumes for producing relatively smaller output volumes. Similarly, DMU7 is the most cost efficient unit which may be intuitively derived from data set in Table 1. According to expectations, the intended conclusion is also derived from the results of proposed method in Table 2. Therefore, results of our method prove these expectations.

In addition, the proposed method is compared with previous methods of cost efficiency evaluation. As noted, the proposed methods in [20, 21, 28] were evaluated cost efficiency of DMUs in a fully fuzzy environment where the input-output data with their corresponding prices are fuzzy. Paryab et al. [20] and Pourmahmoud and Bafekr Sharak [28] apply methods based on the parametric models in which cost efficiency scores depend on the selected parametric values while the proposed method considers a linear model in which the cost efficiency scores no dependence on the parameter values. In the suggested method by Paryab et al., the cost efficiency scores are obtained as crisp numbers while the cost efficiencies by using other methods are as fuzzy numbers. Applying data of Table 1 on the previous methods, it can be seen that DMU5 and DMU7 are introduced as cost efficient units by Puri and Yadav [21]. Paryab et al. introduced DMU4, DMU5 and DMU7 as cost efficient units. Moreover, DMU5 for values of parameter $0.5 \le \alpha \le 1$ and DMU7 for all values of parameter $0 \le \alpha \le 1$ are introduced as fuzzy cost efficient units by Pourmahmoud and Bafekr Sharak [28].

As shown in this section, the proposed method obtained more appropriate interval of fuzzy cost efficiency scores, and easily implement, in compared with previous methods. Moreover, improve strategies are presented for inefficient units in the proposed method, while the other methods cannot.

5. Conclusion

In traditional cost efficiency models, the decision maker must work with a crisp data set which is quite restrictive because input-output data in actual problems are often imprecise. In this study, the extension of traditional cost efficiency models has proposed. This extension applies fuzzy extended multiplication and division to introduce new generalized approach. In this regard, an assumption of fuzzy information has considered that input-output values as well as input prices are supposed as triangular fuzzy numbers. Using the extended operations, a new definition of fuzzy cost efficiency has been proposed. Moreover, the issue of presenting improve strategy was studied for cost inefficient units using proposed method. Finally, a numerical example was presented in order to validate of the proposed approach.

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